Problem 3.39

Find the matrix elements $\langle n|x|n'\rangle$ and $\langle n|p|n'\rangle$ in the (orthonormal) basis of stationary states for the harmonic oscillator (Equation 2.68). You already calculated the "diagonal" elements ($n = n'$) in Problem 2.12; use the same technique for the general case. Construct the corresponding (infinite) matrices, X and P. Show that $(1/2m)\mathsf{P}^2 + (m\omega^2/2)\mathsf{X}^2 = \mathsf{H}$ is *diagonal*, in this basis. Are its diagonal elements what you would expect? *Partial answer:*

$$\langle n\,|\,x\,|\,n'\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n'}\delta_{n,n'-1} + \sqrt{n}\delta_{n',n-1}\right). \tag{3.114}$$

Solution

Use the method of Example 2.5 on page 47 and express the position operator in terms of the promotion and demotion operators, $\hat{a}_+$ and $\hat{a}_-$, respectively.

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}_+ + \hat{a}_-)$$

Representing the nth eigenstate $\psi_n(x)$ of the harmonic oscillator as a ket $|n\rangle$, the promotion and demotion operators satisfy

$$\hat{a}_+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$
$$\hat{a}_-|n\rangle = \sqrt{n}\,|n-1\rangle.$$

So then

$$\begin{aligned}
\langle n\,|\,\hat{x}\,|\,n'\rangle &= (\langle n|\hat{x}\rangle\cdot|n'\rangle \\
&= \left(\hat{x}^\dagger|n\rangle\right)^\dagger\cdot|n'\rangle \\
&= \left[\sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}_+ + \hat{a}_-)^\dagger\,|n\rangle\right]^\dagger\cdot|n'\rangle \\
&= \left[\sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}_+^\dagger + \hat{a}_-^\dagger\right)|n\rangle\right]^\dagger\cdot|n'\rangle \\
&= \left[\sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}_- + \hat{a}_+)\,|n\rangle\right]^\dagger\cdot|n'\rangle \\
&= \left[\sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}_-|n\rangle + \hat{a}_+|n\rangle)\right]^\dagger\cdot|n'\rangle \\
&= \left[\sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,|n-1\rangle + \sqrt{n+1}\,|n+1\rangle\right)\right]^\dagger\cdot|n'\rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,\langle n-1| + \sqrt{n+1}\,\langle n+1|\right)\cdot|n'\rangle.
\end{aligned}$$

Because the eigenstates of the harmonic oscillator are orthonormal, the Kronecker delta symbol appears.

$$\begin{aligned}\langle n\,|\,\hat{x}\,|\,n'\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\,\langle n-1\,|\,n'\rangle + \sqrt{n+1}\,\langle n+1\,|\,n'\rangle\right] \\ &= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,\delta_{n-1,n'} + \sqrt{n+1}\,\delta_{n+1,n'}\right)\end{aligned}$$

The first term in parentheses is nonzero when $n-1=n'$, or $n=n'+1$. The second term in parentheses is nonzero when $n+1=n'$, or $n'=n+1$. Therefore,

$$\boxed{\langle n\,|\,\hat{x}\,|\,n'\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,\delta_{n,n'+1} + \sqrt{n'}\,\delta_{n',n+1}\right).}$$

X is the matrix representation for the position operator in the basis of eigenstates for the harmonic oscillator. The element at row n and column n' (both starting from 0) is $\langle n\,|\,\hat{x}\,|\,n'\rangle$ as illustrated below.

$$\mathsf{X} = \begin{pmatrix} \langle 0\,|\,\hat{x}\,|\,0\rangle & \langle 0\,|\,\hat{x}\,|\,1\rangle & \langle 0\,|\,\hat{x}\,|\,2\rangle & \cdots & \langle 0\,|\,\hat{x}\,|\,n'\rangle & \cdots \\ \langle 1\,|\,\hat{x}\,|\,0\rangle & \langle 1\,|\,\hat{x}\,|\,1\rangle & \langle 1\,|\,\hat{x}\,|\,2\rangle & \cdots & \langle 1\,|\,\hat{x}\,|\,n'\rangle & \cdots \\ \langle 2\,|\,\hat{x}\,|\,0\rangle & \langle 2\,|\,\hat{x}\,|\,1\rangle & \langle 2\,|\,\hat{x}\,|\,2\rangle & \cdots & \langle 2\,|\,\hat{x}\,|\,n'\rangle & \cdots \\ \vdots & \vdots & \vdots & \ddots & & \\ \langle n\,|\,\hat{x}\,|\,0\rangle & \langle n\,|\,\hat{x}\,|\,1\rangle & \langle n\,|\,\hat{x}\,|\,2\rangle & & \langle n\,|\,\hat{x}\,|\,n'\rangle & \\ \vdots & \vdots & \vdots & & & \ddots \end{pmatrix}$$

The elements for which the row is the column plus one have a value the square root of the row, and the elements for which the column is the row plus one have a value the square root of the column.

$$\langle n\,|\,\hat{x}\,|\,n'\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left({\color{blue}\sqrt{n}\,\delta_{n,n'+1}} + {\color{red}\sqrt{n'}\,\delta_{n',n+1}}\right)$$

Therefore,

$$\mathsf{X} = \sqrt{\frac{\hbar}{2m\omega}}\begin{pmatrix} 0 & {\color{red}\sqrt{1}} & 0 & 0 & 0 & 0 & \cdots \\ {\color{blue}\sqrt{1}} & 0 & {\color{red}\sqrt{2}} & 0 & 0 & 0 & \cdots \\ 0 & {\color{blue}\sqrt{2}} & 0 & {\color{red}\sqrt{3}} & 0 & 0 & \cdots \\ 0 & 0 & {\color{blue}\sqrt{3}} & 0 & {\color{red}\sqrt{4}} & 0 & \cdots \\ 0 & 0 & 0 & {\color{blue}\sqrt{4}} & 0 & {\color{red}\sqrt{5}} & \cdots \\ 0 & 0 & 0 & 0 & {\color{blue}\sqrt{5}} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Now express the momentum operator in terms of the promotion and demotion operators, $\hat{a}_+$ and $\hat{a}_-$, respectively.

$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}\,(\hat{a}_+ - \hat{a}_-)$$

So then

$$\begin{aligned}
\langle n \,|\, \hat{p} \,|\, n' \rangle &= (\langle n|\hat{p}) \cdot |n'\rangle \\
&= \left(\hat{p}^\dagger |n\rangle\right)^\dagger \cdot |n'\rangle \\
&= \left[-i\sqrt{\frac{\hbar m\omega}{2}}\,(\hat{a}_+ - \hat{a}_-)^\dagger\,|n\rangle\right]^\dagger \cdot |n'\rangle \\
&= \left[-i\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a}_+^\dagger - \hat{a}_-^\dagger\right)|n\rangle\right]^\dagger \cdot |n'\rangle \\
&= \left[-i\sqrt{\frac{\hbar m\omega}{2}}\,(\hat{a}_- - \hat{a}_+)\,|n\rangle\right]^\dagger \cdot |n'\rangle \\
&= \left[-i\sqrt{\frac{\hbar m\omega}{2}}\,(\hat{a}_-|n\rangle - \hat{a}_+|n\rangle)\right]^\dagger \cdot |n'\rangle \\
&= \left[-i\sqrt{\frac{\hbar m\omega}{2}}\left(\sqrt{n}\,|n-1\rangle - \sqrt{n+1}\,|n+1\rangle\right)\right]^\dagger \cdot |n'\rangle \\
&= i\sqrt{\frac{\hbar m\omega}{2}}\left(\sqrt{n}\,\langle n-1| - \sqrt{n+1}\,\langle n+1|\right) \cdot |n'\rangle \\
&= i\sqrt{\frac{\hbar m\omega}{2}}\left(\sqrt{n}\,\langle n-1\,|\,n'\rangle - \sqrt{n+1}\,\langle n+1\,|\,n'\rangle\right) \\
&= i\sqrt{\frac{\hbar m\omega}{2}}\left(\sqrt{n}\,\delta_{n-1,n'} - \sqrt{n+1}\,\delta_{n+1,n'}\right).
\end{aligned}$$

The first term in parentheses is nonzero when $n-1=n'$, or $n=n'+1$. The second term in parentheses is nonzero when $n+1=n'$, or $n'=n+1$. Therefore,

$$\boxed{\langle n \,|\, \hat{p} \,|\, n' \rangle = i\sqrt{\frac{\hbar m\omega}{2}}\left(\sqrt{n}\,\delta_{n,n'+1} - \sqrt{n'}\,\delta_{n',n+1}\right).}$$

P is the matrix representation for the momentum operator in the basis of eigenstates for the harmonic oscillator. The element at row n and column n' (both starting from 0) is $\langle n \,|\, \hat{p} \,|\, n' \rangle$ as illustrated below.

$$\mathsf{P} = \begin{pmatrix} \langle 0 \,|\, \hat{p} \,|\, 0 \rangle & \langle 0 \,|\, \hat{p} \,|\, 1 \rangle & \langle 0 \,|\, \hat{p} \,|\, 2 \rangle & \cdots & \langle 0 \,|\, \hat{p} \,|\, n' \rangle & \cdots \\ \langle 1 \,|\, \hat{p} \,|\, 0 \rangle & \langle 1 \,|\, \hat{p} \,|\, 1 \rangle & \langle 1 \,|\, \hat{p} \,|\, 2 \rangle & \cdots & \langle 1 \,|\, \hat{p} \,|\, n' \rangle & \cdots \\ \langle 2 \,|\, \hat{p} \,|\, 0 \rangle & \langle 2 \,|\, \hat{p} \,|\, 1 \rangle & \langle 2 \,|\, \hat{p} \,|\, 2 \rangle & \cdots & \langle 2 \,|\, \hat{p} \,|\, n' \rangle & \cdots \\ \vdots & \vdots & \vdots & \ddots & & \\ \langle n \,|\, \hat{p} \,|\, 0 \rangle & \langle n \,|\, \hat{p} \,|\, 1 \rangle & \langle n \,|\, \hat{p} \,|\, 2 \rangle & & \langle n \,|\, \hat{p} \,|\, n' \rangle & \\ \vdots & \vdots & \vdots & & & \ddots \end{pmatrix}$$

The elements for which the row is the column plus one have a value the square root of the row, and the elements for which the column is the row plus one have a value negative the square root of the column.

$$\langle n \,|\, \hat{p} \,|\, n' \rangle = i\sqrt{\frac{\hbar m\omega}{2}} \left(\sqrt{n}\, \delta_{n,n'+1} - \sqrt{n'}\, \delta_{n',n+1} \right)$$

Therefore,

$$\mathsf{P} = i\sqrt{\frac{\hbar m\omega}{2}} \begin{pmatrix} 0 & -\sqrt{1} & 0 & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & -\sqrt{2} & 0 & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & -\sqrt{4} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & -\sqrt{5} & \cdots \\ 0 & 0 & 0 & 0 & \sqrt{5} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

H is the matrix representation for the Hamiltonian operator in the basis of eigenstates for the harmonic oscillator and is given by

$$\mathsf{H} = \frac{1}{2m}\mathsf{P}^2 + \frac{1}{2}m\omega^2\mathsf{X}^2.$$

Find the matrix elements of X^2 now.

$$\begin{aligned} \langle n \,|\, \hat{x}^2 \,|\, n' \rangle &= \langle n \,|\, \hat{x}\hat{x} \,|\, n' \rangle \\ &= \langle n \,|\, \hat{x}\hat{I}\hat{x} \,|\, n' \rangle \\ &= \langle n \,|\, \hat{x} \left(\sum_{k=0}^{\infty} |k\rangle\langle k| \right) \hat{x} \,|\, n' \rangle \\ &= \sum_{k=0}^{\infty} \langle n \,|\, \hat{x} \,|\, k \rangle \langle k \,|\, \hat{x} \,|\, n' \rangle \end{aligned}$$

Substitute the boxed formula for $\langle n \,|\, \hat{x} \,|\, n' \rangle$ twice.

$$\begin{aligned}
\langle n \,|\, \hat{x}^2 \,|\, n' \rangle &= \sum_{k=0}^{\infty} \left[\sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n}\, \delta_{n,k+1} + \sqrt{k}\, \delta_{k,n+1} \right) \right] \left[\sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{k}\, \delta_{k,n'+1} + \sqrt{n'}\, \delta_{n',k+1} \right) \right] \\
&= \frac{\hbar}{2m\omega} \sum_{k=0}^{\infty} \left(\sqrt{n}\, \delta_{n,k+1} + \sqrt{k}\, \delta_{k,n+1} \right) \left(\sqrt{k}\, \delta_{k,n'+1} + \sqrt{n'}\, \delta_{n',k+1} \right) \\
&= \frac{\hbar}{2m\omega} \sum_{k=0}^{\infty} \left(\sqrt{nk}\, \delta_{n,k+1}\delta_{k,n'+1} + \sqrt{nn'}\, \delta_{n,k+1}\delta_{n',k+1} + k\, \delta_{k,n+1}\delta_{k,n'+1} + \sqrt{kn'}\, \delta_{k,n+1}\delta_{n',k+1} \right) \\
&= \frac{\hbar}{2m\omega} \sum_{k=0}^{\infty} \left(\sqrt{nk}\, \delta_{n-1,k}\delta_{k,n'+1} + \sqrt{nn'}\, \delta_{n-1,k}\delta_{n'-1,k} + k\, \delta_{k,n+1}\delta_{k,n'+1} + \sqrt{kn'}\, \delta_{k,n+1}\delta_{n'-1,k} \right) \\
&= \frac{\hbar}{2m\omega} \left(\sum_{k=0}^{\infty} \sqrt{nk}\, \delta_{n-1,k}\delta_{k,n'+1} + \sqrt{nn'} \sum_{k=0}^{\infty} \delta_{n-1,k}\delta_{k,n'-1} + \sum_{k=0}^{\infty} k\, \delta_{n+1,k}\delta_{k,n'+1} + \sum_{k=0}^{\infty} \sqrt{kn'}\, \delta_{n+1,k}\delta_{k,n'-1} \right) \\
&= \frac{\hbar}{2m\omega} \left[\sqrt{n(n-1)}\, \delta_{n-1,n'+1} + \sqrt{nn'}\, \delta_{n-1,n'-1} + (n+1)\, \delta_{n+1,n'+1} + \sqrt{(n'-1)n'}\, \delta_{n+1,n'-1} \right]
\end{aligned}$$

This first term is nonzero when $k = n - 1$ and $k = n' + 1$, or $n = n' + 2$. The second term is nonzero when $k = n - 1$ and $k = n' - 1$, or $n = n'$. The third term is nonzero when $k = n + 1$ and $k = n' + 1$, or $n = n'$. The fourth term is nonzero when $k = n + 1$ and $k = n' - 1$, or $n' = n + 2$.

$$\mathsf{X}^2 = \frac{\hbar}{2m\omega} \begin{pmatrix}
\sqrt{0(0)} + (0+1) & 0 & \sqrt{(2-1)2} & 0 & 0 & 0 & \cdots \\
0 & \sqrt{1(1)} + (1+1) & 0 & \sqrt{(3-1)3} & 0 & 0 & \cdots \\
\sqrt{2(2-1)} & 0 & \sqrt{2(2)} + (2+1) & 0 & \sqrt{(4-1)4} & 0 & \cdots \\
0 & \sqrt{3(3-1)} & 0 & \sqrt{3(3)} + (3+1) & 0 & \sqrt{(5-1)5} & \cdots \\
0 & 0 & \sqrt{4(4-1)} & 0 & \sqrt{4(4)} + (4+1) & 0 & \cdots \\
0 & 0 & 0 & \sqrt{5(5-1)} & 0 & \sqrt{5(5)} + (5+1) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

Therefore,

$$\mathsf{X}^2 = \frac{\hbar}{2m\omega}\begin{pmatrix} 1 & 0 & \sqrt{2} & 0 & 0 & 0 & \cdots \\ 0 & 3 & 0 & \sqrt{6} & 0 & 0 & \cdots \\ \sqrt{2} & 0 & 5 & 0 & \sqrt{12} & 0 & \cdots \\ 0 & \sqrt{6} & 0 & 7 & 0 & \sqrt{20} & \cdots \\ 0 & 0 & \sqrt{12} & 0 & 9 & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{20} & 0 & 11 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Now find the matrix elements of P^2, substituting the boxed formula for $\langle n \,|\, \hat{p} \,|\, n' \rangle$ twice.

$$\begin{aligned}
\langle n \,|\, \hat{p}^2 \,|\, n' \rangle &= \langle n \,|\, \hat{p}\hat{p} \,|\, n' \rangle \\
&= \langle n \,|\, \hat{p}\hat{I}\hat{p} \,|\, n' \rangle \\
&= \langle n \,|\, \hat{p} \left(\sum_{k=0}^{\infty} |k\rangle\langle k| \right) \hat{p} \,|\, n' \rangle \\
&= \sum_{k=0}^{\infty} \langle n \,|\, \hat{p} \,|\, k \rangle \langle k \,|\, \hat{p} \,|\, n' \rangle \\
&= \sum_{k=0}^{\infty} \left[i\sqrt{\frac{\hbar m\omega}{2}} \left(\sqrt{n}\,\delta_{n,k+1} - \sqrt{k}\,\delta_{k,n+1} \right) \right] \left[i\sqrt{\frac{\hbar m\omega}{2}} \left(\sqrt{k}\,\delta_{k,n'+1} - \sqrt{n'}\,\delta_{n',k+1} \right) \right] \\
&= -\frac{\hbar m\omega}{2} \sum_{k=0}^{\infty} \left(\sqrt{n}\,\delta_{n,k+1} - \sqrt{k}\,\delta_{k,n+1} \right) \left(\sqrt{k}\,\delta_{k,n'+1} - \sqrt{n'}\,\delta_{n',k+1} \right) \\
&= -\frac{\hbar m\omega}{2} \sum_{k=0}^{\infty} \left(\sqrt{nk}\,\delta_{n,k+1}\delta_{k,n'+1} - \sqrt{nn'}\,\delta_{n,k+1}\delta_{n',k+1} - k\,\delta_{k,n+1}\delta_{k,n'+1} + \sqrt{kn'}\,\delta_{k,n+1}\delta_{n',k+1} \right) \\
&= -\frac{\hbar m\omega}{2} \sum_{k=0}^{\infty} \left(\sqrt{nk}\,\delta_{n-1,k}\delta_{k,n'+1} - \sqrt{nn'}\,\delta_{n-1,k}\delta_{n'-1,k} - k\,\delta_{k,n+1}\delta_{k,n'+1} + \sqrt{kn'}\,\delta_{k,n+1}\delta_{n'-1,k} \right) \\
&= -\frac{\hbar m\omega}{2} \left(\sum_{k=0}^{\infty} \sqrt{nk}\,\delta_{n-1,k}\delta_{k,n'+1} - \sqrt{nn'} \sum_{k=0}^{\infty} \delta_{n-1,k}\delta_{k,n'-1} - \sum_{k=0}^{\infty} k\,\delta_{n+1,k}\delta_{k,n'+1} + \sum_{k=0}^{\infty} \sqrt{kn'}\,\delta_{n+1,k}\delta_{k,n'-1} \right) \\
&= -\frac{\hbar m\omega}{2} \left[\sqrt{n(n-1)}\,\delta_{n-1,n'+1} - \sqrt{nn'}\,\delta_{n-1,n'-1} - (n+1)\,\delta_{n+1,n'+1} + \sqrt{(n'-1)n'}\,\delta_{n+1,n'-1} \right] \\
&= -\frac{\hbar m\omega}{2} \left[\sqrt{n(n-1)}\,\delta_{n,n'+2} - \sqrt{nn'}\,\delta_{n,n'} - (n+1)\,\delta_{n,n'} + \sqrt{(n'-1)n'}\,\delta_{n+2,n'} \right]
\end{aligned}$$

This first term is nonzero when $k = n - 1$ and $k = n' + 1$, or $n = n' + 2$. The second term is nonzero when $k = n - 1$ and $k = n' - 1$, or $n = n'$. The third term is nonzero when $k = n + 1$ and $k = n' + 1$, or $n = n'$. The fourth term is nonzero when $k = n + 1$ and $k = n' - 1$, or $n' = n + 2$.

Therefore,

$$\mathsf{P}^2 = -\frac{\hbar m\omega}{2}\begin{pmatrix} -\sqrt{0(0)}-(0+1) & 0 & \sqrt{(2-1)2} & 0 & 0 & 0 & \cdots \\ 0 & -\sqrt{1(1)}-(1+1) & 0 & \sqrt{(3-1)3} & 0 & 0 & \cdots \\ \sqrt{2(2-1)} & 0 & -\sqrt{2(2)}-(2+1) & 0 & \sqrt{(4-1)4} & 0 & \cdots \\ 0 & \sqrt{3(3-1)} & 0 & -\sqrt{3(3)}-(3+1) & 0 & \sqrt{(5-1)5} & \cdots \\ 0 & 0 & \sqrt{4(4-1)} & 0 & -\sqrt{4(4)}-(4+1) & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{5(5-1)} & 0 & -\sqrt{5(5)}-(5+1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= -\frac{\hbar m\omega}{2}\begin{pmatrix} -1 & 0 & \sqrt{2} & 0 & 0 & 0 & \cdots \\ 0 & -3 & 0 & \sqrt{6} & 0 & 0 & \cdots \\ \sqrt{2} & 0 & -5 & 0 & \sqrt{12} & 0 & \cdots \\ 0 & \sqrt{6} & 0 & -7 & 0 & \sqrt{20} & \cdots \\ 0 & 0 & \sqrt{12} & 0 & -9 & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{20} & 0 & -11 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Now that X^2 and P^2 are known, the Hamiltonian matrix can be calculated.

$$\begin{aligned}
\mathsf{H} &= \frac{1}{2m}\mathsf{P}^2 + \frac{1}{2}m\omega^2\mathsf{X}^2 \\
&= -\frac{1}{2m}\frac{\hbar m\omega}{2}\begin{pmatrix} -1 & 0 & \sqrt{2} & 0 & 0 & 0 & \cdots \\ 0 & -3 & 0 & \sqrt{6} & 0 & 0 & \cdots \\ \sqrt{2} & 0 & -5 & 0 & \sqrt{12} & 0 & \cdots \\ 0 & \sqrt{6} & 0 & -7 & 0 & \sqrt{20} & \cdots \\ 0 & 0 & \sqrt{12} & 0 & -9 & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{20} & 0 & -11 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \frac{1}{2}m\omega^2\frac{\hbar}{2m\omega}\begin{pmatrix} 1 & 0 & \sqrt{2} & 0 & 0 & 0 & \cdots \\ 0 & 3 & 0 & \sqrt{6} & 0 & 0 & \cdots \\ \sqrt{2} & 0 & 5 & 0 & \sqrt{12} & 0 & \cdots \\ 0 & \sqrt{6} & 0 & 7 & 0 & \sqrt{20} & \cdots \\ 0 & 0 & \sqrt{12} & 0 & 9 & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{20} & 0 & 11 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\
&= -\frac{\hbar\omega}{4}\begin{pmatrix} -1 & 0 & \sqrt{2} & 0 & 0 & 0 & \cdots \\ 0 & -3 & 0 & \sqrt{6} & 0 & 0 & \cdots \\ \sqrt{2} & 0 & -5 & 0 & \sqrt{12} & 0 & \cdots \\ 0 & \sqrt{6} & 0 & -7 & 0 & \sqrt{20} & \cdots \\ 0 & 0 & \sqrt{12} & 0 & -9 & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{20} & 0 & -11 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \frac{\hbar\omega}{4}\begin{pmatrix} 1 & 0 & \sqrt{2} & 0 & 0 & 0 & \cdots \\ 0 & 3 & 0 & \sqrt{6} & 0 & 0 & \cdots \\ \sqrt{2} & 0 & 5 & 0 & \sqrt{12} & 0 & \cdots \\ 0 & \sqrt{6} & 0 & 7 & 0 & \sqrt{20} & \cdots \\ 0 & 0 & \sqrt{12} & 0 & 9 & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{20} & 0 & 11 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}
\end{aligned}$$

Therefore,

$$\mathsf{H} = \frac{\hbar\omega}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 3 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 5 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 7 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 9 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 11 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The matrix is diagonal because the eigenenergies of the harmonic oscillator are along the main diagonal, and all other elements are zero.